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REGULARITY PROPERTIES OF IDEALS AND ULTRAFILTERS

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For an arbitrary ideal I on the regular cardinal κ we consider the problem of refining a given collection $\{A_\alpha : \alpha<\kappa\}\subseteq \mathcal{P}(\kappa)-I$ by another collection $\{B_\alpha : \alpha<\kappa\}\subseteq\mathcal{P}(\kappa)-I$ so that the sets in the latter collection are as nearly pairwise disjoint as possible. In this context we discuss regularity of ultrafilters, saturation of ideals and some problems of Fodor and Ulam.

0. Introduction

We begin by establishing some notation. Throughout this paper μ will denote an infinite cardinal, κ will always denote an uncountable *regular* cardinal, and we will use the phrase "ideal on κ" to mean "proper uniform ideal on κ". That is, to say I is an ideal on κ means that I is a collection of subsets of κ that is closed under the taking of subsets and finite unions and, moreover, $[\kappa]^{<\kappa}\subseteq I\neq\mathcal{P}(\kappa)$. If I is an ideal on κ, then I^+ denotes $\mathcal{P}(\kappa)-I$ (the sets of "positive I-measure") and $I^*=\{X\subseteq\kappa : \kappa-X\in I\}$ (the sets of "I-measure one"). If $A\in I^+$, then the restriction of I to A is the ideal $I\,|\,A=\{X\subseteq\kappa : X\cap A\in I\}$.

We use NS_κ to denote the (normal) ideal of non-stationary subsets of κ. If X is a set of ordinals, then $\mathrm{OT}(X)$ denotes the order type of X. If $\mathcal{A}=\{A_\alpha : \alpha<\lambda\}$ and $\mathcal{B}=\{B_\alpha : \alpha<\lambda\}$ are two collections of sets indexed by an ordinal λ, then $\mathcal{B}$ is called a *refinement of* $\mathcal{A}$ iff $B_\alpha\subseteq A_\alpha$ for every $\alpha<\lambda$. The following notion of regularity of an ideal provides a simultaneous generalization of some problems dealing with uniform ultrafilters on κ and κ-complete ideals on κ.

Definition 0.1. (i) Suppose $(\lambda_1,\lambda_2,\lambda_3)$ is a triple of ordinals and I is an ideal on κ. If $\mathcal{A}=\{A_\alpha : \alpha<\lambda_3\}\subseteq I^+$, then a refinement $\mathcal{B}=\{B_\alpha : \alpha<\lambda_3\}$ is called an *I-$(\lambda_1,\lambda_2,\lambda_3)$-refinement of* $\mathcal{A}$ iff $\mathcal{B}\subseteq I^+$ and $(*)$ holds:

$$\text{If}\quad X\subseteq\lambda_3\quad\text{and}\quad \mathrm{OT}(X)\geq\lambda_2,\quad\text{then}\quad \mathrm{OT}(\textstyle\bigcap\{B_\alpha : \alpha\in X\})\leq\lambda_1 \qquad (*)$$

(ii) An ideal I on κ is called *$(\lambda_1,\lambda_2,\lambda_3)$-regular* iff every λ_3 indexed collection $\mathcal{A}=\{A_\alpha : \alpha<\lambda_3\}\subseteq I^+$ has an I-$(\lambda_1,\lambda_2,\lambda_3)$-refinement.

(iii) An ideal I on κ is called *regular* iff it is $(0,\omega,\kappa)$-regular.

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Notice that if I is a maximal ideal on κ (i e I^* is a uniform ultrafilter on κ), then I is $(\lambda_1, \lambda_2, \lambda_3)$-regular iff there exists a collection $\mathcal{A} = \{A_\alpha \ \alpha < \lambda_3\} \subseteq I^*$ satisfying ($\dagger$) of Definition 0 1(i) Hence the following definitions are equivalent to the usual ones (see [5, 11])

Definition 0.2. If $\mathcal{U}$ is a uniform ultrafilter on μ^+ and $\mathcal{U} = I^+$, then $\mathcal{U}$ is called

(i) (μ, μ^+)-*regular* iff I is $(0, \mu, \mu^+)$-regular,
(ii) *weakly* (μ, μ^+)-*regular* iff I is (μ, μ, μ^+)-regular,
(iii) $(\mu+1, \mu^+)$-*regular* iff I is $(0, \mu+1, \mu^+)$-regular,
(iv) *regular* iff I is regular

Part of the motivation for Definition 0 1 comes from Fodor's question (see [2]) asking whether or not NS_{ω_1} is $(0, 2, \omega_1)$-regular (I e can every collection of ω_1 stationary subsets of ω_1 be refined by a collection of pairwise disjoint stationary sets?) Baumgartner et al [2] show that this is really a question about the degree of saturation of NS_{ω_1} In particular, they show that an affirmative answer to Fodor's question lies in strength between Ulam's theorem [21] that NS_{ω_1} is nowhere ω_1-saturated and the conjecture that NS_{ω_1} is nowhere ω_2-saturated

Definition 0.3. (i) If I is a κ-complete ideal on κ, then we will say that I satisfies *Fodor's property* iff I is $(0, 2, \kappa)$-regular

It would seem that our definition of regular ideal is more geared to the consideration of ultrafilters than of κ-complete ideals on κ That is, while $(0, \omega, \kappa)$-regularity is the most one can expect from ultrafilters, it might seem more natural to reserve the phrase "regular κ-complete ideal on κ" for those ideals satisfying Fodor's property We show in Section 5 that we can have it both ways That is, the regular κ-complete ideals on κ are precisely the ones satisfying Fodor's property Hence, Definition 0 1 gives a simultaneous generalization of a regularity notion for uniform ultrafilters on μ^+ and a saturation notion for μ^+-complete ideals on μ^+ It is well-known that analogies persist with respect to consequences of and arguments pertaining to saturated ideals on the one hand and non-regular ultrafilters on the other Much of our approach here is to use Definition 0 1 to both explore and to exploit these analogies

In Section 1 we consider a simultaneous generalization of the notions of "normal ideal" and "weakly normal ultrafilter" and we establish some weak regularity results in this context that will be needed later In Section 2 we combine these with some known results in order to prove that every uniform ultrafilter on μ^+ is both $(\mu+1, \mu^+)$-regular (answering a question of Benda [5]) and weakly (μ, μ^+)-regular (answering a question of Prikry)

Section 3 contains a number of results about saturated ideals While these results are needed in Sections 4 through 7, they seem to be of some interest in their own right For example, we show that every ω_2-saturated ω_1-complete ideal on ω_1 is the intersection of countably many isomorphs of normal ω_2-saturated ideals on ω_1.

Section 4 is the "κ-complete analogue" of Section 2, and we show here that every μ^+-complete ideal on μ^+ is both $(0, \mu+1, \mu^+)$-regular and $(1, \mu, \mu^+)$-regular. In analogy with Kanamori's theorem on regular ultrafilters, we also show that if μ is singular then every μ^+-complete ideal on μ^+ is $(0, \mu, \mu^+)$-regular.

As indicated above, Section 5 deals with the relationship between regular ideals and ideals satisfying Fodor's property.

An old question of Ulam [7, Problem 81] asks if there can exist a collection $\mathcal{I}$ of ω_1-complete ideals on ω_1 such that $|\mathcal{I}| = \omega_1$ and for every $X \subseteq \omega_1$ there is some $I \in \mathcal{I}$ such that $X \in I \cup I^*$. A consequence of the results in Section 6 is the equivalence between the existence of such a family consisting entirely of normal ideals and the existence of an ω_1-complete non-regular ideal on ω_1.

In Section 7 we show that there is a non-regular ω_1-complete ideal on ω_1 iff there is an ω_1-complete ideal I on ω_1 such that $\mathcal{P}(\omega_1)/I$ has a dense set of size ω_1. This result (strengthening a theorem of Baumgartner et al. [2]) was noticed independently by Balcar and Vojtáš [1]. In this section we also show that $MA_{\aleph_1}$ implies that all ω_1-complete ideals on ω_1 are regular. The analogous result for ultrafilters is due to Laver. Hence $MA_{\aleph_1}$ settles Fodor's problem and the "normal version" of Ulam's problem.

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1. Weakly normal ideals

Definition 1.1. An ideal I on κ will be called *weakly normal* iff every function f that is regressive on a set of positive I-measure is bounded on a set of positive I-measure.

Notice that if I is a κ-complete ideal on κ, then I is weakly normal iff I is normal, and if I is maximal, then I is weakly normal iff I^* is a weakly normal ultrafilter in the usual sense (see [10, 11]). It is easy to see that if I is a weakly normal ideal on κ, then $\{\alpha \in \kappa : \alpha \text{ is a successor ordinal}\} \in I$. In fact, it turns out that $NS_\kappa \subseteq I$ whenever I is a weakly normal ideal on κ.

The following theorem about weakly normal ideals will allow us to immediatly derive the desired weak regularity results for weakly normal ultrafilters and normal ideals. The idea behind the proof has been used by Ketonen [13] and others in the context of ultrafilters and Galvin [2] and others in the context of

κ-complete ideals on κ. We are grateful to Fred Galvin for bringing this basic idea to our attention.

Theorem 1.2. *Suppose I is a weakly normal ideal on κ and $\{S_\alpha : \alpha<\kappa\} \subseteq I^+$ is given. Then there exists a refinement $\{A_\alpha : \alpha<\kappa\} \subseteq I^+$ of $\{S_\alpha : \alpha<\kappa\}$ such that (1) and (2) hold.*

(1) *If $X \subseteq \kappa$ and $\xi \in \bigcap\{A_\alpha : \alpha \in X\}$, then* $\mathrm{OT}(X) \leq \mathrm{cf}(\xi)$.

(2) *If $\xi_1, \xi_2 \in \bigcap\{A_\alpha : \alpha \in X\}$ and* $\mathrm{OT}(X) = \mathrm{cf}(\xi_1) = \mathrm{cf}(\xi_2)$, *then $\xi_1 = \xi_2$.*

Proof. Let $B = \{\xi<\kappa : \xi$ is a limit ordinal$\}$ and notice that $B \in I^*$ since I is weakly normal. For each $\xi \in B$ let $f_\xi : \mathrm{cf}(\xi) \to \xi$ be a strictly increasing function whose range is cofinal in ξ. We define sets $\{A_\alpha : \alpha<\kappa\}$ and ordinals $\{\mu_\alpha : \alpha<\kappa\}$ simultaneously by induction. Suppose A_β and μ_β have been defined for all $\beta<\alpha$ and suppose that $\mu_\beta<\kappa$ for every $\beta<\alpha$. Let $\mu'_\alpha = \sup\{\mu_\beta : \beta<\alpha\}$ and let $A'_\alpha = S_\alpha \cap B - (\mu'_\alpha + 1)$. Then $\mu'_\alpha < \kappa$ so $A'_\alpha \in I^+$. Define $g_\alpha : A'_\alpha \to \kappa$ by

$$g_\alpha(\xi) = \inf\{f_\xi(\delta) : \mu'_\alpha < f_\xi(\delta)\}.$$

Then g_α is regressive on A'_α so there exists a set $A_\alpha \in I^+$ and an ordinal μ_α such that $A_\alpha \subseteq A'_\alpha$ and $\mu'_\alpha < \mu_\alpha < \kappa$ and $g_\alpha(A_\alpha) \subseteq \mu_\alpha$. Hence, for every $\xi \in A_\alpha$ there exists $\delta_\xi \in \mathrm{cf}(\xi)$ such that $\mu'_\alpha < f_\xi(\delta_\xi) < \mu_\alpha$. This defines A_α and μ_α and completes the construction of $\{A_\alpha : \alpha<\kappa\}$ and $\{\mu_\alpha : \alpha<\kappa\}$.

To see that property (1) is satisfied, suppose $X \subseteq \kappa$ and $\xi \in \bigcap\{A_\alpha : \alpha \in X\}$. Consider the function $h_\xi : X \to \mathrm{range}(f_\xi)$ where

$$h_\xi(\alpha) = g_\alpha(\xi).$$

Notice first that if $\alpha \in X$, then $\xi \in A_\alpha \subseteq A'_\alpha$ so $g_\alpha(\xi)$ is defined. Moreover, h_ξ is an order preserving map, since if $\alpha_1, \alpha_2 \in X$ and $\alpha_1 < \alpha_2$, then

$$h_\xi(\alpha_1) = g_{\alpha_1}(\xi) < \mu_{\alpha_1} \leq \mu'_{\alpha_2} < g_{\alpha_2}(\xi) = h_\xi(\alpha_2).$$

Hence $\mathrm{OT}(X) \leq \mathrm{cf}(\xi)$.

Finally, to prove (2) we show that if $\mathrm{OT}(X) = \mathrm{cf}(\xi)$ and $\xi \in \bigcap\{A_\alpha : \alpha \in X\}$, then $\xi = \sup\{\mu_\alpha : \alpha \in X\}$. As above we have that if $\alpha_1, \alpha_2 \in X$ and $\alpha_1 < \alpha_2$, then $h_\xi(\alpha_1) < \mu_{\alpha_1} < h_\xi(\alpha_2) < \mu_{\alpha_2}$, and h_ξ is an order preserving map from X to the range of f_ξ. Hence $\xi = \sup(\mathrm{range}(f_\xi)) = \sup(h_\xi(X)) = \sup\{\mu_\alpha : \alpha \in X\}$. Assertion (2) now follows immediately.

Corollary 1.3. *If I is a weakly normal ideal on κ and $\{\xi<\kappa : \mathrm{cf}(\xi) \leq \mu\} \in I^*$, then I is $(0, \mu+1, \kappa)$-regular and I is $(1, \mu, \kappa)$-regular.*

Corollary 1.4. *If I is a weakly normal ideal on κ and $\{\xi<\kappa : \mathrm{cf}(\xi) < \mu\} \in I^*$, then I is $(0, \mu, \kappa)$-regular.*

2. Weak regularity of ultrafilters

In this section we combine the results of Section 1 with some known results of Kanamori and Ketonen in order to show that every uniform ultrafilter on μ^+ is both $(\mu+1, \mu^+)$-regular and weakly (μ, μ^+)-regular.

If $\mathcal{U}$ is a uniform ultrafilter on κ, then a function $f: \kappa \to \kappa$ is said to be *bounded* (mod $\mathcal{U}$) provided that $\{\xi < \kappa: f(\xi) < \alpha\} \in \mathcal{U}$ for some $\alpha < \kappa$. Otherwise, f is called *unbounded* (mod $\mathcal{U}$), and in this case the uniform ultrafilter $f_*(\mathcal{U})$ is defined by

$$X \in f_*(\mathcal{U}) \quad \text{iff} \quad f^{-1}(X) \in \mathcal{U}.$$

The uniform ultrafilter $\mathcal{U}$ on κ is called a *P-point* iff every function $f: \kappa \to \kappa$ is either bounded (mod $\mathcal{U}$) or less than κ to 1 on a set $A \in \mathcal{U}$ (i.e. $|f^{-1}(\alpha) \cap A| < \kappa$ for every $\alpha < \kappa$). $\mathcal{U}$ is a weakly normal ultrafilter iff $\mathcal{U}$ is uniform and every regressive function $f: \kappa \to \kappa$ is bounded (mod $\mathcal{U}$). In particular then, $\mathcal{U}$ is a weakly normal ultrafilter on κ iff $\mathcal{U}^* = I$ is a weakly normal ideal (in the sense of Definition 1.1). Hence the results of Section 1 allow us to immediately derive the following result, at least part of which may well have been noticed independently by others.

Theorem 2.1. *If $\mathcal{U}$ is a weakly normal ultrafilter on μ^+ and $\mathcal{U} = I^*$, then I is both $(0, \mu+1, \mu^+)$-regular and $(1, \mu, \mu^+)$-regular.*

In order to derive results from the above for uniform ultrafilters on μ^+ that are not weakly normal, we need the following well-known results of Ketonen and Kanamori together with one easy lemma (which we state in somewhat more generality than is actually required for our present purposes).

Theorem 2.2. (1) (Ketonen [5]) *Suppose $\mathcal{U}$ is a uniform ultrafilter on μ^+ that is not (μ, μ^+)-regular. Then for every $f: \mu^+ \to \mu^+$ that is unbounded* (mod $\mathcal{U}$) *there exists a set $X \in \mathcal{U}$ such that $f \mid X$ is less than μ to 1. In particular, $\mathcal{U}$ is a P-point ultrafilter.*

(2) (Kanamori [10]) *Suppose $\mathcal{U}$ is a uniform ultrafilter on μ^+ that is not (μ, μ^+)-regular. Then there exists a function $f: \mu^+ \to \mu^+$ such that $f_*(\mathcal{U})$ is weakly normal.*

Lemma 2.3. *Suppose $\mathcal{U}$ is a uniform ultrafilter on κ that is not $(\lambda_1, \lambda_2, \lambda_3)$-regular. Suppose $f: \kappa \to \kappa$ is unbounded* (mod $\mathcal{U}$) *and* $\mathrm{OT}(f(X)) > \lambda_1$ *whenever* $\mathrm{OT}(X) > \lambda_1$ *and $X \subseteq \kappa$. Then $f_*(\mathcal{U})$ is not $(\lambda_1, \lambda_2, \lambda_3)$-regular.*

We omit the easy proof of Lemma 2.3 and state now the weak regularity results for ultrafilters that follow from the above.

Theorem 2.4. *For any uniform ultrafilter $\mathcal{U}$ on the successor cardinal $\kappa = \mu^+$ the following hold:*

(1) *$\mathcal{U}$ is $(\mu+1, \mu^+)$-regular.*

(2) *There exists $\{A_\alpha: \alpha < \mu^+\} \subseteq \mathcal{U}$ such that $|\bigcap\{A_\alpha: \alpha \in X\}| < \mu$ whenever $|X| \geq \mu$.*

(3) *$\mathcal{U}$ is weakly (μ, μ^+)-regular.*

Assertion (1) answers a question of Benda [5] and assertion (3) answers a question of Prikry, who obtained (3) under the assumption that $2^\kappa = \kappa^+$ (see [9, 11]).

Theorem 2.4(2) shows that for any uniform ultrafilter $\mathscr{U}$ on ω_1 there is a collection $\{A_\alpha : \alpha<\omega_1\}\subseteq\mathscr{U}$ such that $\bigcap\{A_\alpha : \alpha\in X\}$ is finite whenever X is an infinite subset of ω_1. The following example (provided by the referee) shows that any extension of this result would require proving that every uniform ultrafilter on ω_1 is regular.

Let $\mathscr{U}$ be a uniform ultrafilter on ω_1 that is not (ω,ω_1)-regular. Since $\mathscr{U}$ is not countably complete there exists a pairwise disjoint partition $\{A_n : n\in\omega\}$ of ω_1 into sets not in $\mathscr{U}$; let $\{B_\alpha : \alpha<\omega_1\}$ be a partition of ω_1 into pairwise disjoint sets such that $|B_\alpha|=n$ if $\alpha\in A_n$. Let

$$F=\{X\subseteq\omega_1 : \exists n\{\alpha<\omega_1 : |B_\alpha-X|\leq n\}\in\mathscr{U}\}$$

Then F is a uniform (proper) filter on ω_1 and so can be extended to a uniform ultrafilter $\mathscr{U}'$ which is easily seen not to be (n,ω,ω_1)-regular for any $n\in\omega$.

The referee also notes the following limitation on improving Theorem 2.4(2) for cardinals $\kappa=\mu^+>\omega_1$. Suppose G.C.H. holds and $\mathscr{U}$ is a uniform ultrafilter on μ^+ which is not (μ,μ^+)-regular. If $\lambda<\mu$ and $\mathscr{U}'$ is any uniform ultrafilter on λ, then the product $\mathscr{U}\times\mathscr{U}'$ is not (γ,μ,μ^+)-regular for any $\gamma<\lambda$.

Ketonen has shown [13] that if $\mathscr{U}$ is a weakly normal ultrafilter on κ and μ is a cardinal less than κ, then $\mathscr{U}$ is (μ,κ)-regular iff $\{\alpha<\kappa : \mathrm{cf}(\alpha)<\mu\}\in\mathscr{U}$. From this and Theorem 1.2 we get the following generalization of Theorem 2.1.

Theorem 2.5. *Suppose μ is a cardinal less than κ and $\mathscr{U}$ is a weakly normal (μ^+,κ)-regular ultrafilter on κ. Then $\mathscr{U}$ is both $(\mu+1,\kappa)$-regular and $(1,\mu,\kappa)$-regular.*

To conclude this section we note that Magidor has recently shown [16] that if the existence of a huge cardinal is consistent, then it is consistent that there is a uniform ultrafilter on ω_2 that is not (ω,ω_2)-regular. Whether or not there can be one on ω_2 that is not (ω_1,ω_2)-regular is open.

3. Saturated ideals

Our goal in this section and the next is to show that every μ^+-complete ideal I on μ^+ is both $(0,\mu+1,\mu^+)$-regular and $(1,\mu,\mu^+)$-regular. Our approach will roughly parallel that of Section 2, where the analogous results for uniform ultrafilters on μ^+ were established. That is, we use the results of Section 1 to obtain the conclusion for normal ideals on μ^+ and then we reduce the general case to this one. For uniform ultrafilters on μ^+, this reduction was made possible by results of Ketonen and Kanamori. For κ-complete ideals on κ, however, we must prove some theorems about saturated ideals in order to make this reduction go through. This is our primary motivation for the considerations of this section, although the results obtained here seem to be of some interest in their own right.

Our starting point is to generalize some of the ultrafilter notions introduced at the beginning of Section 2 to the context of κ-complete ideals on κ. Suppose then that I is a κ-complete ideal on κ. A function $f: \kappa \to \kappa$ will be called *unbounded* (mod I) provided that $f^{-1}(\alpha) \in I$ for every $\alpha < \kappa$. If f is unbounded (mod I), then the κ-complete ideal $f_*(I)$ is defined by

$$X \in f_*(I) \quad \text{iff} \quad f^{-1}(X) \in I.$$

If J is also a κ-complete ideal on κ, then I and J are said to be *isomorphic* (denoted $I \cong J$) iff $J = f_*(I)$ for some bijection $f: \kappa \to \kappa$. If $I \cong J$ and J is some normal ideal on κ, then I is called a *normal isomorph*.

The κ-complete ideal I on κ will be called a *P-point* provided that every $f: \kappa \to \kappa$ is either constant on a set of positive I-measure or less than κ to 1 on a set of I-measure one. In particular then, if I is a P-point and f is unbounded (mod I), then there exists a set $X \in I^+$ such that $|f^{-1}(\alpha) \cap X| < \kappa$ for every $\alpha < \kappa$. It should be noted that our definition of "P-point ideal" is different than the notion used by Kanamori in [10].

The following result is the analogue of Ketonen's theorem 2.2, and it is, of course, directly inspired by the work of Benda and Ketonen [5].

Theorem 3.1. *Suppose κ is a successor cardinal and I is a κ-complete κ^+-saturated ideal on κ. Then I is a P-point ideal.*

Proof. Let $\kappa = \mu^+$ and suppose $f: \kappa \to \kappa$ shows that I is not a P-point. Then $f^{-1}(\alpha) \in I$ for every $\alpha < \kappa$ and for every $X \in I^+$ we have that $|X \cap f^{-1}(\alpha)| = \kappa$ for some $\alpha < \kappa$.

Claim. *If $g: \kappa \to \kappa$, then the set $B_g = \{\xi < \kappa: g \circ f(\xi) < \xi\} \in I^+$.*

Proof of claim. Suppose $B_g \in I$ and let

$$C_g = \{\xi < \kappa: g \circ f(\xi) \geq \xi\}.$$

Then $C_g = \kappa - B_g$ so $C_g \in I^+$. Hence, there is some $\alpha < \kappa$ such that $|C_g \cap f^{-1}(\alpha)| = \kappa$, so we can choose $\xi \in C_g \cap f^{-1}(\alpha)$ such that $\xi > g(\alpha)$. But then $g \circ f(\xi) = g(f(\xi)) = g(\alpha) < \xi$ so $\xi \notin C_g$. This proves the claim.

Let $\{g_\alpha: \alpha < \kappa^+\}$ be a collection of eventually different functions mapping κ to κ. *That is, if* $\alpha < \beta < \kappa^+$, then $|\mathrm{Agree}(g_\alpha, g_\beta)| < \kappa$ where $\mathrm{Agree}(g_\alpha, g_\beta) = \{\xi < \kappa: g_\alpha(\xi) = g_\beta(\xi)\}$. Such a collection always exists.

For each $\xi < \kappa$ let $\gamma_\xi: \xi \to \mu$ be one to one. (Recall that $\kappa = \mu^+$.) For each $\alpha < \kappa^+$ define $h_\alpha: B_{g_\alpha} \to \mu$ by

$$h_\alpha(\xi) = \gamma_\xi(g_\alpha(f(\xi)))$$

Then $B_{g_\alpha} \in I^+$ (by the claim) and if $\xi \in B_{g_\alpha}$, then $g_\alpha(f(\xi)) < \xi$ so $h_\alpha(\xi)$ is well-defined and less than μ.

Since I is κ-complete it is possible to choose, for each $\alpha<\kappa^+$, a set $X_\alpha \subseteq B_{g_\alpha}$ and an ordinal $\mu_\alpha<\mu$ such that $X_\alpha \in I^+$ and $h_\alpha(X_\alpha)=\{\mu_\alpha\}$ Now choose $Y\subseteq \kappa^+$ and $\lambda<\mu$ such that $|Y|=\kappa^+$ and $\mu_\alpha=\lambda$ for all $\alpha\in Y$

We claim that $\{X_\alpha : \alpha\in Y\}$ shows that I is not κ^+-saturated Suppose, for contradiction, that $X_\alpha \cap X_\beta \in I^+$ for some $\{\alpha,\beta\}\in[Y]^2$ Since $|\{\delta<\kappa : X_\alpha\cap X_\beta \cap f^{-1}(\delta)\neq 0\}|=\kappa$ we can choose $\xi\in X_\alpha\cap X_\beta$ such that $f(\xi)>$ sup Agree (g_α, g_β) Then $h_\alpha(\xi)=\lambda=h_\beta(\xi)$ so $\gamma_\xi(g_\alpha(f(\xi)))=\gamma_\xi(g_\beta(f(\xi)))$ But γ_ξ is one to one, so $g_\alpha(f(\xi))=g_\beta(f(\xi))$ This contradicts the fact that $f(\xi)>$ sup Agree (g_α, g_β) and completes the proof

If I is a κ-complete ideal on κ, then a function $f : \kappa\to\kappa$ is called *incompressible for I* provided that f is unbounded (mod I) and for every $g : \kappa\to\kappa$, if

$$\{\xi<\kappa : g(\xi)<f(\xi)\}\in I^+,$$

then g is constant on a set of positive I-measure The importance of this notion lies in the following

Theorem 3.2 (Solovay [18]) *If I is a κ-complete κ^+-saturated ideal on κ, then there exists a function $f : \kappa\to\kappa$ that is incompressible for I and in this case $f_*(I)$ is a normal κ^+-saturated ideal on κ*

Solovay indicates in [18] two ways to prove Theorem 3 2, one using Boolean ultrapowers and one being purely combinatorial We have another combinatorial proof of this that we plan to include in [4] It is important for our purposes to notice that for the situation described in Theorem 3 2 not only is $f_*(I)$ a normal κ^+-saturated ideal on κ but in fact $f_*(I \mid A)$ is a normal κ^+-saturated ideal on κ for every $A\in I^+$

Theorems 3 1 and 3 2 yield the following characterization theorem for saturated ideals on successor cardinals Its present form owes much to James Baumgartner and Stan Wagon

Theorem 3.3. *Suppose $\kappa=\mu^+$ and I is an ideal on κ Then I is κ^+-saturated and κ-complete iff there exists $\lambda\leqslant\mu$ and a pairwise disjoint partition $\{A_\alpha : \alpha<\lambda\}$ of κ such that for each $\alpha<\lambda$ $I\mid A_\alpha$ is isomorphic to a normal κ^+-saturated ideal on κ*

Proof. The right to left direction is easy and left to the reader For the left to right direction, note that by Solovay's Theorem 3 2 (and the remark in the paragraph after it) there exists a function $f : \kappa\to\kappa$ such that $f_*(I\mid A)$ is a normal κ^+-saturated ideal on κ for every set $A\in I^+$ By Theorem 3 1 there exists a set $X\in I^*$ such that $|f^{-1}(\alpha)\cap X|<\kappa$ for every $\alpha<\kappa$ Let $B_\alpha=f^{-1}(\alpha)\cap X$ and let $\mu_\alpha=|B_\alpha|$ For each $\alpha<\kappa$ let $\{b^\xi_\alpha : \xi<\mu_\alpha\}$ enumerate B_α, and for each $\xi<\mu$ let $A_\xi=\{b^\xi_\alpha : \alpha<\kappa\}$ Then $\bigcup\{A_\xi : \xi<\mu\}\in I^*$, $A_{\xi_1}\cap A_{\xi_2}=0$ if $\xi_1\neq\xi_2$, and $f\mid A_\xi$ is one to one for each $\xi<\mu$ Thus, for each $\xi<\mu$ either $A_\xi \in I$ or else $I \mid A_\xi \cong f_*(I \mid A_\xi)$ and $f_*(I \mid A_\xi)$ is a normal κ^+-saturated ideal on κ

Corollary 3.4. *If κ is a successor cardinal and I is a κ^+-saturated κ-complete ideal on κ, then I can be written as the intersection of fewer than κ normal isomorphs*

Proof. Let $\{A_\alpha \ \ \alpha<\lambda\}$ be as in Theorem 3 3 and for each $\alpha<\lambda$ let $N_\alpha = I \mid A_\alpha$ Then it is easy to see that $I=\bigcap\{N_\alpha \ \ \alpha<\lambda\}$

Several people independently noticed that Theorem 3 3 is best possible in the sense that "$\lambda \leqslant \mu$" cannot be replaced by "$\lambda<\mu$" Perhaps the cleanest description of an example showing this is the following (provided by Kunen and included with his permission) Let I be a κ^+-saturated κ-complete ideal on $\kappa=\mu^+$ and define an ideal J on $\mu\times\kappa$ by

$$J=\{X\subseteq \mu\times\kappa \ \ \forall\alpha<\mu\{\beta<\kappa \ \ (\alpha,\beta)\in X\}\in I\}$$

Let π be the projection $\pi \ \ \mu\times\kappa\to\kappa$ If $\{A_\alpha \ \ \alpha<\lambda\}$ is a partition of $\mu\times\kappa$ such that $J \mid A_\alpha$ is a normal isomorph, then $\pi \mid (A_\alpha\cap X_\alpha)$ is one to one for some $X_\alpha\in J^*$ Let $X=\bigcap\{X_\alpha \ \ \alpha<\lambda\}$ Then $X\in J^*$ and $\pi \mid X$ is λ to 1 Since $X\in J^*$, $\bigcap\{X(\alpha) \ \ \alpha<\mu\}\in I^*$ where $X(\alpha)=\{\beta<\kappa \ \ (\alpha,\beta)\in X\}$ But now if $\beta\in\bigcap\{X(\alpha) \ \ \alpha<\mu\}$, then $|\pi^{-1}(\beta)|=\mu$ so $\lambda\geqslant\mu$ Since J is clearly κ^+-saturated, this shows $\lambda\leqslant\mu$ in Theorem 3 3 is best possible

A consequence of a result of Weglorz [22] is that every normal ideal on κ is a P-point Several arguments involving normal ideals make use of the fact that if I is a normal ideal on κ and $\{A_\alpha \ \ \alpha<\kappa\}\subseteq I$, then there exists $A\in I$ such that $|A_\alpha - A|<\kappa$ for every $\alpha<\kappa$ (i e let A be the diagonal union of the sequence $\{A_\alpha \ \ \alpha<\kappa\}$) It is not difficult to see that this same property holds for every κ-complete P-point ideal on κ For this reason, many results stated for normal ideals actually hold for all P-point ideals, and hence, if $\kappa=\mu^+$, then for all κ^+-saturated κ-complete ideals on κ As an example of this, we state a result from [3] (which we will need in Sections 5 and 7 anyway) and show how it can be strengthened via Theorem 3 3

Theorem 3.5 (see [3]) *Let I be a normal ideal on κ Then I is κ^+-saturated iff every normal ideal on κ extending I is of the form $I \mid A$ for some $A\in I^+$*

By combining Theorem 3 1 with the proof of Theorem 3 5 (which occurs as Theorem 3 1 of [3]), it is easy to see that one obtains the following.

Theorem 3.6. *Suppose I is a κ^+-saturated κ-complete ideal on the successor cardinal κ and suppose J is a κ-complete ideal on κ extending I Then the following are equivalent conditions on J*

(1) *J is κ^+-saturated*

(2) *J is a P-point*

(3) *$J=I \mid A$ for some $A\in I^+$*

We conclude this section with an application of Theorem 3 3 to the number of ω_2-saturated ideals on ω_1 Kunen has shown [14] that if the existence of a huge cardinal is consistent, then it is consistent that there is an ω_2-saturated (normal) ideal on ω_1 (Even stronger results have recently been obtained by Laver [15]) It is still open whether or not NS_{ω_1} can be ω_2-saturated

Theorem 3.7. *If* NS_{ω_1} *is* ω_2*-saturated, then there are exactly* $2^{\aleph_1}$ ω_2*-saturated* ω_1*-complete ideals on* ω_1

Proof. Let $\{S_\alpha \ \alpha<\omega_1\}$ be a partition of ω_1 into pairwise disjoint stationary sets Let $\mathscr{Y}=\{A\subseteq\omega_1 \ A\neq 0$ and $A\neq\omega_1\}$ and for each $A\in\mathscr{Y}$ let $T_A=\bigcup\{S_\alpha \ \alpha\in A\}$ It is not hard to check that if NS_{ω_1} is ω_2-saturated, then $\{\mathrm{NS}_{\omega_1}\mid T_A \ A\in\mathscr{Y}\}$ is a collection of $2^{\aleph_1}$ ω_2-saturated ω_1-complete ideals (It turns out that they are also pairwise non isomorphic) Now by Theorem 3 5, if NS_{ω_1} is ω_1-saturated, then every normal ω_2-saturated ideal on ω_1 is of the form $\mathrm{NS}_{\omega_1}\mid A$ for some $A\subseteq\omega_1$ Hence there are only $2^{\aleph_1}$ normal ω_2-saturated ideals on ω_1 and thus only $2^{\aleph_1}\cdot 2^{\aleph_1}=2^{\aleph_1}$ isomorphs of normal ω_2-saturated ideals on ω_1. Thus, by Theorem 3 3 there are only $(2^{\aleph_1})^{\aleph_0}=2^{\aleph_1}$ ω_2-saturated ω_1-complete ideals on ω_1

4. Weak regularity of κ-complete ideals on κ

In this section we build on work of Baumgartner, Galvin, Hajnal and Máté in order to prove the following

Theorem 4.1. *If I is a μ^+-complete ideal on μ^+, then I is both $(0,\mu+1,\mu^+)$-regular and $(1,\mu,\mu^+)$-regular If μ is singular, then I is in fact $(0,\mu,\mu^+)$-regular*

Notice that the third part of Theorem 4 1 is the analogue of Kanamori's theorem [10] which asserts that if μ is singular, then every uniform ultrafilter on μ^+ is $(\mu\ \mu^+)$-regular Our starting point in the proof of Theorem 4 1 is to apply Theorem 1 2 to immediately obtain the desired result in the special case where I is a normal ideal on μ^+ The first part of Lemma 4 2 is due entirely to Galvin and is included with his permission The second part was noticed independently by Galvin and myself

Lemma 4.2 (Galvin) *If I is a normal ideal on μ^+, then I is both $(0,\mu+1,\mu^+)$-regular and $(1\ \mu,\mu^+)$-regular If μ is singular, then I is in fact $(0,\mu,\mu^+)$-regular*

Proof. This follows immediately from Corollaries 1 3 and 1 4

The next lemma is due to Baumgartner, Hajnal and Máté This is the result we were referring to in the introduction when we remarked that these "regularity of ideals" properties were actually "saturation properties of ideals". This will be further illuminated in later sections.

Lemma 4.3 (Baumgartner et al. [2]). *Suppose I is a κ-complete ideal on κ and $\mathscr{A}=\{A_\alpha : \alpha<\kappa\}\subseteq I^+$ is such that $I \mid A_\alpha$ is not κ^+-saturated for any $\alpha<\kappa$. Then there exists an I-$(0,2,\kappa)$-refinement of $\mathscr{A}$. Hence, if I is nowhere κ^+-saturated, then I is $(0,2,\kappa)$-regular (i.e. I satisfies Fodor's property).*

The next lemma is not really needed in this section, since we only appeal to the special case of it that we have isolated below as Lemma 4.5. In later sections, we will need the added generality built into the statement of Lemma 4.4.

Lemma 4.4 *Suppose I is a κ-complete ideal on κ and suppose $\{A_\alpha : \alpha<\kappa\}\subseteq I^+$ satisfies the following:*

> *For every $\alpha<\kappa$ there exists a set $B_\alpha\in\mathscr{P}(A_\alpha)\cap I^+$ such that there is an I-$(\lambda_1,\lambda_2,\kappa)$-refinement for $\{C^\alpha_\xi : \xi<\kappa\}$ where*
>
> $$C^\alpha_\xi=\begin{cases}B_\alpha\cap B_\xi & \text{if} \quad B_\alpha\cap B_\xi\in I^+,\\ B_\alpha & \text{otherwise}\end{cases} \tag{*}$$

Then $\{A_\alpha : \alpha<\kappa\}$ has an I-$(\lambda_1,\lambda_2,\kappa)$-refinement.

Proof. For each $\alpha<\kappa$ let B_α be as guaranteed to exist by (*), and let $\{D^\alpha_\xi : \xi<\kappa\}$ be the I-$(\lambda_1,\lambda_2,\kappa)$-refinement for $\{C^\alpha_\xi : \xi<\kappa\}$. Define $g: \kappa\to\kappa$ by

$$g(\alpha)=\inf\{\beta<\kappa: B_\beta\cap B_\alpha\in I^+\}.$$

Notice that $g(\alpha)\leqslant\alpha$. Let $T_\alpha=D^{g(\alpha)}_\alpha-\bigcup\{B_\xi : \xi<g(\alpha)\}$. We claim that $\{T_\alpha : \alpha<\kappa\}$ is the desired I-$(\lambda_1,\lambda_2,\kappa)$-refinement of $\{A_\alpha : \alpha<\kappa\}$.

Notice first that $D^{g(\alpha)}_\alpha\in I^+$ and $D^{g(\alpha)}_\alpha\subseteq C^{g(\alpha)}_\alpha=B_\alpha\cap B_{g(\alpha)}\subseteq A_\alpha$. Also, if $\xi<g(\alpha)$, then $B_\alpha\cap B_\xi\in I$ so T_α is the result of subtracting off fewer than κ sets in I from a set of positive I-measure. Hence $T_\alpha\subseteq A_\alpha$ and $T_\alpha\in I^+$.

Next, notice that if $g(\alpha)\neq g(\beta)$, then $T_\alpha\cap T_\beta=0$. That is, if $g(\alpha)<g(\beta)$, then $T_\alpha\subseteq B_{g(\alpha)}$. Also, if $\xi<g(\beta)$, then $T_\beta\cap B_\xi=0$, so $T_\beta\cap B_{g(\alpha)}=0$. Thus $T_\alpha\cap T_\beta=0$.

Now suppose $X\subseteq\kappa$, $\mathrm{OT}(X)\leqslant\lambda_2$ and $\mathrm{OT}(\bigcap\{T_\xi : \xi\in X\})>\lambda_1$. Then there exists an $\alpha<\kappa$ such that $g(X)=\{\alpha\}$. Then for all $\xi\in X$ we have $T_\xi\subseteq D^\alpha_\xi$. Hence $\mathrm{OT}(\bigcap\{D^\alpha_\xi : \xi\in X\})>\lambda_1$ contradicting the fact that $\{D^\alpha_\xi : \xi<\kappa\}$ is an I-$(\lambda_1,\lambda_2,\kappa)$-refinement of $\{C^\alpha_\xi : \xi<\kappa\}$.

Notice that in Lemma 4.4 if there exists a refinement $\{B_\alpha : \alpha<\kappa\}\subseteq I^+$ for $\{A_\alpha : \alpha<\kappa\}$ such that each $I \mid B_\alpha$ is $(\lambda_1,\lambda_2,\kappa)$-regular for each $\alpha<\kappa$, then (*) is trivially satisfied. Hence, an immediate consequence of 4.4 is the following.

Lemma 4.5. *Suppose I is a κ-complete ideal on κ and suppose $\{A_\alpha\ \alpha<\kappa\}\subseteq I^+$ has a refinement $\{B_\alpha\ \alpha<\kappa\}\subseteq I^+$ such that for each $\alpha<\kappa$ $I\mid B_\alpha$ is $(\lambda_1,\lambda_2,\kappa)$-regular Then $\{A_\alpha\ \alpha<\kappa\}$ has an I-$(\lambda_1,\lambda_2,\kappa)$ refinement*

Theorem 4 1 is an immediate consequence of Lemma 4 2 and the following result

Theorem 4.6. *Suppose $\kappa=\mu^+$ and every normal ideal on κ is $(\lambda_1,\lambda_2,\kappa)$-regular Then every κ-complete ideal on κ is $(\lambda_1,\lambda_2,\kappa)$-regular*

Proof. Let I be a κ-complete ideal on $\kappa=\mu^+$ and suppose $\{A_\alpha\ \alpha<\kappa\}\subseteq I^+$ is given We show that $\{A_\alpha\ \alpha<\kappa\}$ has an I-$(\lambda_1,\lambda_2,\kappa)$-refinement by appealing to Lemma 4 5 Fix $\alpha<\kappa$ and let $J=I\mid A_\alpha$

Case 1 J is nowhere κ^+-saturated

Then, by Lemma 4 3 J is $(0,2,\kappa)$-regular, so if we set $B_\alpha=A_\alpha$ then $I\mid B_\alpha$ is $(\lambda_1,\lambda_2,\kappa)$-regular

Case 2 $J\mid A$ is κ^+-saturated for some $A\in J^+$

Without loss of generality, assume that $A\subseteq A_\alpha$ so $J\mid A=I\mid A$ By Theorem 3 3 there exists a set $B\in(I\mid A)^+$ such that $I\mid A\mid B$ is isomorphic to a normal ideal N on κ Let $B_\alpha=A\cap B$ Then $B_\alpha\in\mathcal{P}(A_\alpha)\cap I^+$ and $I\mid B_\alpha\cong N$ By assumption, N is $(\lambda_1,\lambda_2,\kappa)$-regular and so it easily follows that $I\mid B_\alpha$ is $(\lambda_1,\lambda_2,\kappa)$-regular In fact, if λ_1 is a cardinal, then it is obvious that $(\lambda_1,\lambda_2,\kappa)$-regularity is preserved by isomorphisms If λ_1 is not a cardinal, then one needs to use the fact that if $f\ \kappa\to\kappa$ is a bijection, then $f\mid C$ is order preserving for some set $C\in(\mathrm{NS}_\kappa)^*\subseteq N^*$

$$(\text{I e let}\quad C=\{\alpha<\kappa\ \ f(\alpha)\geqslant\alpha\quad\text{and}\quad\forall\beta<\alpha(f(\beta)<\alpha)\}\ )$$

Corollary 4.7. *Suppose $\kappa=\mu^+$ and there is a non-regular κ-complete ideal on κ Then there is a normal non-regular ideal on κ*

Proof. This follows immediately from Theorem 4 6 with $(\lambda_1,\lambda_2,\kappa)=(0,\omega,\kappa)$

Having completed the proof of Theorem 4 1, it is worth remarking that there is a direct way to show that every μ^+-complete ideal on μ^+ is $(1,\mu,\mu^+)$-regular This involves using a slightly "suped up" version of an Ulam matrix in the obvious way The usual construction of an Ulam matrix $M=\{M(\xi,\alpha)\ \ \xi<\mu,\alpha<\mu^+\}$ on μ^+ involves choosing, for each $\alpha<\mu^+$, a bijection $f_\alpha\ \ |\alpha|\to\alpha$ and then letting $M(\xi,\alpha)=\{\beta<\alpha\ \ f_\beta(\xi)=\alpha\}$ What we want to do is to choose the maps f_α more carefully so that if $\alpha<\beta<\mu^+$, then $|\{\eta\ \ f_\alpha(\eta)=f_\beta(\eta)\}|<\mu$ This can be done by an inductive construction that defines f_α by a "back and forth" argument. It turns out that the resulting Ulam matrix M then has, in addition to the usual properties, the property that if $X\in[\mu\times\mu^+]^\mu$, then $|\bigcap\{M(p)\ \ p\in X\}|\leqslant 1$ We leave the details here for the reader

5. Fodor's property and non-regular ideals

In the introduction we said that an ideal I on κ would be called regular iff it is $(0, \omega, \kappa)$-regular, and we said that if I was $(0, 2, \kappa)$-regular, then we would say I satisfies Fodor's property. The former definition is motivated by ultrafilter considerations, while it seems likely that the latter concept is probably the most natural notion of "regularity" for a κ-complete ideal on a successor cardinal κ. In this section we resolve this disparity of terminology by proving that a κ-complete ideal I on κ is regular iff it satisfies Fodor's property. For the case where I is a normal ideal, this result is essentially due to Laver. In a preliminary version of this paper, we used this fact together with the techniques employed in previous sections to eliminate the normality assumption for the case where κ is a successor cardinal. The result in its present form, and the following elegant proof, were provided by the referee.

Theorem 5.1. *Suppose I is a κ-complete ideal on κ. Then I is regular iff I satisfies Fodor's property.*

Proof. Only the right to left direction requires proof, so suppose that I is a regular κ-complete ideal on κ, and let $\{A_\alpha : \alpha < \kappa\} \subseteq I^+$ be given. Since I is regular we lose no generality in assuming that $\bigcap\{A_\alpha : \alpha \in X\} = 0$ for every infinite $X \subseteq \kappa$. For each $\beta < \kappa$ we define $\mathcal{Q}(\beta)$ (the "occurrences of β") by

$$\mathcal{Q}(\beta) = \{\xi < \kappa : \beta \in A_\xi\}.$$

Then $\mathcal{Q}(\beta)$ is finite, so let s_β be the finite sequence listing the elements of $\mathcal{Q}(\beta)$ in increasing order. If s and t are finite sequences we will say that s extends t iff $s \mid m = t$ for some $m \leqslant \text{length}(s)$, and we say s property extends t iff s extends t but $s \neq t$.

For any finite increasing sequence s of ordinals less than κ and any $n \in \omega$ let $A(s, n) \subseteq \kappa$ be defined by

$$A(s, n) = \{\beta < \kappa : \text{length}(s_\beta) = n \text{ and } s_\beta \text{ extends } s\}.$$

Call s n-maximal iff $A(s, n) \in I^+$ but $A(t, n) \in I$ for every proper extension t of s. Notice the following:

(i) if $A(s, n) \neq 0$, then $\text{length}(s) \leqslant n$,

(ii) if t extends s, then $A(t, n) \subseteq A(s, n)$,

(iii) if $A(s, n) \in I^+$, then there is an n-maximal t such that t extends s,

(iv) if t is n-maximal and t' is n'-maximal and $A(t, n) \cap A(t', n') \neq 0$, then $n = n'$ and $t = t'$.

Both (i) and (ii) are obvious and (iii) easily follows from (i). For (iv), suppose $\beta \in A(t, n) \cap A(t', n')$. Then $n = \text{length}(s_\beta) = n'$ and s_β extends both t and t'. Thus, if t and t' are not equal, then either t is a proper extension of t' or t' is a proper extension of t But this contradicts (respectively) the n'-maximality of t' or the n-maximality of t

Now, if s is n-maximal and length $(s)=k$, we let $\{B_i(s,n)\ i<k\}$ be a pairwise disjoint partition of $A(s,n)$ into sets in I^+ This is possible since the regularity of I guarantees that $I\mid A$ is not a prime ideal for any $A\in I^+$

Fix $\xi<\kappa$ Using the κ-completeness of I it is easy to find a set $B_\xi\subseteq A_\xi$, a finite sequence s and integers i and n such that $B_\xi\in I^+$ and for every $\beta\in B_\xi$ length $(s_\beta)=n$ and $s_\beta\mid(i+1)=s$ and $s(i)=\xi$ Notice that $B_\xi\subseteq A(s,n)\subseteq A_\xi$ and so we can assume that $B_\xi=A(s,n)$ Hence, by (i) above, there is an n-maximal t such that t extends s Let $C_\xi=B_i(t,n)$

To complete the proof it suffices to show that $\{C_\xi\ \xi<\kappa\}$ is an I-$(0,2,\kappa)$-refinement of $\{A_\xi\ \xi<\kappa\}$ Notice first that $C_\xi=B_i(t,n)\subseteq A(t,n)\subseteq A(s,n)=B_\xi\subseteq A_\xi$ Now, suppose that $\xi<\xi'<\kappa$ and assume, for contradiction, that $\beta\in C_\xi\cap C_{\xi'}$ Let $C_\xi=B_i(t,n)$ and $C_{\xi'}=B_{i'}(t',n')$ Then $\beta\in A(t,n)\cap A(t',n')$ so by (iv) above $n=n'$ and $t=t'$ Hence $\beta\in B_i(t,n)\cap B_{i'}(t,n)$ so $i=i'$ But now $\xi=s_\beta(i)=\xi'$ which is the desired contradiction

Recall that in Theorem 4 6 we proved that if κ is a successor cardinal, then all κ-complete ideals on κ are $(\lambda_1,\lambda_2,\kappa)$-regular iff all normal ideals on κ are $(\lambda_1,\lambda_2,\kappa)$-regular Our next goal in this section is to show that the hypothesis that κ be a successor cardinal can be dropped for the special case in which $\lambda_1=0$

Lemma 5.2. *Suppose I is a κ^+-saturated κ-complete ideal on κ and f is the incompressible* (mod I) *function guaranteed to exist by Solovay's theorem* (3 2) *Then for every $B\in I^+$ there exists a set $C\subseteq f(B)$ such that $f_*(I\mid B)=f_*(I)\mid C$*

Proof. Notice first that $f_*(I\mid B)\supseteq f_*(I)\mid f(B)$ That is, if $X\in f_*(I)\mid f(B)$, then $X\cap f(B)\in f_*(I)$ so $f^{-1}(X\cap f(B))\in I$ But $f^{-1}(X)\cap B\subseteq f^{-1}(X)\cap f^{-1}(f(B))=f^{-1}(X\cap f(B))\in I$ Thus $f^{-1}(X)\cap B\in I$ so $X\in f_*(I\mid B)$

Secondly as remarked in the paragraph following Theorem 3 2, $f_*(I\mid B)$ is a normal ideal on κ Also $f_*(I)$ is κ^+-saturated and so $f_*(I)\mid f(B)$ is a normal κ^+-saturated ideal on κ It now follows from Theorem 3 5 that $f_*(I\mid B)=f_*(I)\mid f(B)\mid C$ for some set $C\in(f_*(I)\mid f(B))^+$ But $f_*(I)\mid f(B)\mid C=f_*(I)\mid(f(B)\cap C)$ and we can clearly assume that $C\subseteq f(B)$ Thus $f_*(I\mid B)=f_*(I)\mid C$ as desired

Theorem 5.3. *Suppose that every normal ideal on κ is $(0,\lambda,\kappa)$-regular Then every κ-complete ideal on κ is $(0,\lambda,\kappa)$-regular*

Proof. Let I be a κ-complete ideal on κ and let $\{A_\alpha\ \alpha<\kappa\}\subseteq I^+$ be given To produce the desired I-$(0,\lambda,\kappa)$-regular refinement, we appeal to Lemma 4 5 Fix $\alpha<\kappa$ and let $J=I\mid A_\alpha$

Case 1 *J is nowhere κ^+-saturated*

Then by Lemma 4 3 J is $(0, 2, \kappa)$-regular so if we set $B_\alpha = A_\alpha$, then $I \mid B_\alpha$ is $(0, \lambda, \kappa)$-regular as desired

Case 2 $J \mid B_\alpha$ is κ^+-saturated for some set $B_\alpha \in \mathcal{P}(A_\alpha) \cap I^+$

Let $K = J \mid B_\alpha$ and suppose $\{C_\alpha : \alpha < \kappa\} \subseteq K^+$ is given Let $f: \kappa \to \kappa$ be the incompressible (mod K) function guaranteed to exist by Theorem 3 2 For each $\alpha < \kappa$ we apply Lemma 5 2 to C_α to obtain a set $D_\alpha \subseteq f(C_\alpha)$ so that $f_*(K \mid C_\alpha) = f_*(K) \mid D_\alpha$ Now consider the normal ideal $f_*(K)$ and the collection $\{D_\alpha : \alpha < \kappa\} \subseteq f_*(K)^+$ Let $\{E_\alpha : \alpha < \kappa\}$ be an $f_*(K)$-$(0, \lambda, \kappa)$-refinement of $\{D_\alpha : \alpha < \kappa\}$ and for each $\alpha < \kappa$ let $F_\alpha = f^{-1}(E_\alpha) \cap C_\alpha$ We claim that $\{F_\alpha : \alpha < \kappa\}$ is the desired K-$(0, \lambda, \kappa)$-refinement of $\{C_\alpha : \alpha < \kappa\}$ The important point to check, of course, is that $F_\alpha \in K^+$ But $E_\alpha \in f_*(I) \mid D_\alpha = f_*(I \mid C_\alpha)$ so $F_\alpha = f^{-1}(E_\alpha) \cap C_\alpha \in I^+$ Suppose now that $X \subseteq \kappa$ and $\mathrm{OT}(X) = \lambda$ and $\xi \in \bigcap\{F_\alpha : \alpha \in X\}$ Then $f(\xi) \in E_\alpha$ for all $\alpha \in X$ and this contradicts the fact that $\{F_\alpha : \alpha < \kappa\}$ is an $f_*(K)$-$(0, \lambda, \kappa)$-refinement of $\{D_\alpha : \alpha < \kappa\}$

Hence, in Case 2 we see that $J \mid B_\alpha (= I \mid B_\alpha)$ is $(0, \lambda, \kappa)$-regular and so the theorem follows from Lemma 4 5

Corollary 5.4. *If there is a non-regular κ-complete ideal on κ, then there is a normal ideal on κ that does not satisfy Fodor's property (and hence is also non-regular)*

6. An equivalence involving Ulam's problem

Ulam's problem [7, Problem 81] is the following Can one define $\aleph_1$ σ-additive 0–1 measures on ω_1 so that each subset of ω_1 is measurable with respect to at least one of them? Thus Ulam's problem asks about the "saturation" of a set of ω_1-complete ideals on ω_1, and so it will be useful to extend some of the notation and terminology of ideals to the context of sets of ideals

Definition 6.1. If $\mathcal{I}$ is a set of ideals on κ, then $\mathcal{I}^+$ denotes $\bigcap\{I^+ : I \in \mathcal{I}\}$ $\mathcal{I}$ will be called λ-*saturated* iff there does not exist a collection $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{I}^+$ such that $X_\alpha \cap X_\beta \in \bigcap \mathcal{I}$ for $\alpha < \beta < \lambda$

In terms of Definition 6 1, Ulam's problem asks if there is a 2-saturated set $\mathcal{I}$ of ω_1-complete ideals on ω_1 such that $|\mathcal{I}| = \omega_1$ Alaoglu and Erdos [6] showed that if $\mathcal{I}$ is a set of ω_1-complete ideals on ω_1 and $|\mathcal{I}| < \omega_1$, then $\mathcal{I}$ is not 2-saturated (Their proof actually yields that such an $\mathcal{I}$ is not ω_1-saturated) It turns out that a different proof of their result extends to show that if $\kappa = \mu^+$ and $\mathcal{I}$ is a set of κ-complete ideals on κ such that $|\mathcal{I}| < \kappa$, then $\mathcal{I}$ is not κ-saturated (see [20]) Prikry [17] showed that Kurepa's hypothesis for ω_1 implies that there is no ω_1-saturated set $\mathcal{I}$ of ω_1-complete ideals on ω_1 such that $|\mathcal{I}| = \omega_1$, and it is shown in [20] that in this situation no such $\mathcal{I}$ can even be ω_2-saturated

A natural generalization of Ulam's problem would result from asking if there can exist a 2-saturated set $\mathcal{I}$ of *normal* ideals on ω_1 such that $|\mathcal{I}|=\omega_1$. A consequence of the results in this section and the next is that an affirmative answer to this generalized version of Ulam's problem is equivalent to the existence of a non-regular ω_1-complete ideal on ω_1. (In fact, these are just two of the eight equivalences listed in Theorem 7.7.) In this section, however, we consider an arbitrary (regular) cardinal κ and we relate the saturation of sets of normal ideals on κ to the considerations of the previous sections by means of the following.

Theorem 6.2. *For any regular uncountable cardinal κ the following assertions are equivalent.*

(i) *There is a non-regular κ-complete ideal on κ.*
(ii) *There is a normal non-regular ideal on κ.*
(iii) *Some κ-complete ideal on κ fails to satisfy Fodor's property.*
(iv) *Some normal ideal on κ fails to satisfy Fodor's property.*
(v) *There is a κ-saturated set $\mathcal{I}$ of normal ideals on κ and $|\mathcal{I}|\leq\kappa$.*

Proof. The equivalence of (i)–(iv) is an immediate consequence of Theorems 5.1 and 5.3, so it suffices to show that (iv) is equivalent to (v). The proof that (iv)→(v) is quite easy and goes as follows. Suppose (v) is false. Let I be a normal ideal on κ and suppose $\{A_\alpha : \alpha<\kappa\}\subseteq I^+$ is given. Then $\mathcal{I}=\{I\mid A_\alpha : \alpha<\kappa\}$ is a set of κ normal ideals on κ so $\mathcal{I}$ is not κ-saturated. Hence, there exists $\{X_\alpha : \alpha<\kappa\}\subseteq\mathcal{I}^+$ such that $X_\alpha\cap X_\beta\in\bigcap\mathcal{I}$ for every $\alpha<\beta<\kappa$. For each $\alpha<\kappa$ let $B_\alpha=A_\alpha\cap X_\alpha-\bigcup\{X_\beta : \beta<\alpha\}$. Since $X_\alpha\in(I\mid A_\alpha)^+$ we have that $A_\alpha\cap X_\alpha\in I^+$, and if $\beta<\alpha$, then $X_\beta\cap X_\alpha\in I\mid A_\alpha$. Thus $B_\alpha\in\mathcal{P}(A_\alpha)\cap I^+$ for every $\alpha<\kappa$ so $\{B_\alpha : \alpha<\kappa\}$ is the desired I-$(0, 2, \kappa)$-refinement of $\{A_\alpha : \alpha<\kappa\}$.

The proof that (v)→(iv) is somewhat more involved and requires the following three lemmas, the first of which is inspired by a proof in [2].

Lemma 6.3. *Suppose $\mathcal{I}=\{I_\alpha : \alpha<\kappa\}$ is an indexed set of (not necessarily distinct) normal ideals on κ none of which is κ^+-saturated. Then there exists a pairwise disjoint partition $\{Z_\alpha : \alpha<\kappa\}$ of κ such that $Z_\alpha\in I_\alpha^+$ for each $\alpha<\kappa$.*

Proof. For each $\alpha<\kappa$ let $\mathcal{X}_\alpha=\{X_\beta^\alpha : \beta<\kappa^+\}\subseteq I_\alpha^+$ be such that $|X_{\beta_1}^\alpha\cap X_{\beta_2}^\alpha|<\kappa$ for $\beta_1<\beta_2<\kappa^+$. This is possible since I_α is normal and not κ^+-saturated (e.g. see [2]). Define a function $H:\kappa\to\kappa$ by

$$H(\xi)=\inf\{\alpha\in\kappa : |\{\beta<\kappa^+ : X_\beta^\alpha\in I_\xi^+\}|=\kappa^+\}.$$

Notice that $H(\xi)\leq\xi$ for every $\xi<\kappa$. Choose $\lambda<\kappa^+$ such that for every $\xi<\kappa$, if $\alpha<H(\xi)$ and $X_\beta^\alpha\in I_\xi^+$, then $\beta<\lambda$. For each $\alpha<\kappa$ let $\mathcal{X}'_\alpha=\{X_\beta^\alpha : \lambda<\beta<\kappa^+\}$. It is now easy to inductively construct a sequence $\{Y_\xi : \xi<\kappa\}$ of distinct sets such that $Y_\xi\in\mathcal{X}'_{H(\xi)}\cap I_\xi^+$ for every $\xi<\kappa$. For each $\xi<\kappa$ let $Y'_\xi=Y_\xi-\nabla\{Y_\alpha : H(\alpha)<H(\xi)\}$

where ∇ denotes diagonal union Then $Y'_\xi \in I^+_\xi$ since $Y_\alpha \in I_\xi$ whenever $H(\alpha) < H(\xi)$. It is now easy to see that $\{Y'_\xi\ \xi<\kappa\}$ has the property that $|Y'_{\xi_1} \cap Y'_{\xi_2}| < \kappa$ whenever $\xi_1 \neq \xi_2$. Finally, let

$$Z_\alpha = Y'_\alpha - \bigcup\{Y'_\beta\ \beta<\alpha\}$$

and notice that $\{Z_\alpha\ \alpha<\kappa\}$ is the desired collection

Lemma 6.4. *Suppose $\mathcal{I}$ is a set of nowhere κ^+-saturated normal ideals on κ such that $|\mathcal{I}| \leq \kappa$ Then there exists a set $\{Y_\alpha\ \alpha<\kappa^+\} \subseteq \mathcal{I}^+$ such that $Y_\alpha \cap Y_\beta \in \mathrm{NS}_\kappa$ whenever $\alpha<\beta<\kappa^+$*

Proof. Let $\{I_\alpha\ \alpha<\kappa\}$ be an enumeration (with repititions if necessary) of $\mathcal{I}$ By Lemma 6 3 there exists a pairwise disjoint partition $\{Z_\alpha\ \alpha<\kappa\}$ of κ such that $Z_\alpha \in I^+_\alpha$ for every $\alpha<\kappa$ Without loss of generality assume that $Z_\alpha \cap (\alpha+1) = 0$ For each $\alpha<\kappa$ let $\{Z^\beta_\alpha\ \beta<\kappa^+\}$ be such that $Z^\beta_\alpha \in \mathcal{P}(Z_\alpha) \cap I^+_\alpha$ and $|Z^{\beta_1}_\alpha \cap Z^{\beta_2}_\alpha| < \kappa$ whenever $\beta_1<\beta_2<\kappa^+$. Now for each $\beta<\kappa^+$ let $Y_\beta = \bigcup\{Z^\beta_\alpha\ \alpha<\kappa\}$ Then $Y_\beta \in \mathcal{I}^+$ and if $\beta_1 \neq \beta_2$, then $Y_{\beta_1} \cap Y_{\beta_2} \subseteq \bigcup\{Z^{\beta_1}_\alpha \cap Z^{\beta_2}_\alpha\ \alpha<\kappa\} \in \mathrm{NS}_\kappa$ since $Z^{\beta_1}_\alpha \cap Z^{\beta_2}_\alpha \subseteq Z_\alpha \subseteq \kappa-(\alpha+1)$ and $|Z^{\beta_1}_\alpha \cap Z^{\beta_2}_\alpha| < \kappa$ Thus $\{Y_\alpha\ \alpha<\kappa^+\}$ is the desired collection

Lemma 6.5. *Suppose $\mathcal{I}$ is a set of κ^+-saturated normal ideals on κ such that $|\mathcal{I}| \leq \kappa$ and $\mathcal{I}$ is κ-saturated Then some normal ideal I on κ fails to satisfy Fodor's property*

Proof. Let $\{I_\alpha,\ \alpha<\kappa\}$ be an enumeration (with repititions if necessary) of $\mathcal{I}$, and let $I = \bigcap\{I_\alpha\ \alpha<\kappa\}$ Then I is clearly a normal ideal and we claim that I is κ^+-saturated If not, then there exists a collection $\{X_\alpha\ \alpha<\kappa^+\} \subseteq I^+$ such that $|X_\alpha \cap X_\beta| < \kappa$ for $\alpha \neq \beta$ (because I is normal) But then for each $\alpha<\kappa^+$ there is some $\beta<\kappa$ such that $X_\alpha \in I^+_\beta$, so we get a set $X \subseteq \kappa^+$ such that $|X| = \kappa^+$ and $\{X_\alpha\ \alpha \in X\}$ shows that some I_β is not κ^+-saturated

Thus I is a normal κ^+-saturated ideal and each $I_\alpha \in \mathcal{I}$ is a normal ideal extending I Hence, it follows from Theorem 3 5 that there is a set $\{A_\alpha\ \alpha<\kappa\} \subseteq I^+$ such that for each $\alpha<\kappa$ $I_\alpha = I \mid A_\alpha$

Suppose, for contradiction, that I satisfies Fodor's property Then there exists an I-$(0,2,\kappa)$-refinement $\{B_\alpha\ \alpha<\kappa\}$ of $\{A_\alpha\ \alpha<\kappa\}$ If $I \mid B_\alpha$ is κ-saturated for some $\alpha<\kappa$, then clearly I fails to satisfy Fodor's property and we are done Suppose then that each $I \mid B_\alpha$ is not κ-saturated and let $\{B^\beta_\alpha\ \beta<\kappa\}$ be a pairwise disjoint partition of β_α into sets in I^+ For each $\beta<\kappa$ let $C_\beta = \bigcup\{B^\beta_\alpha\ \alpha<\kappa\}$. Then $\{C_\beta\ \beta<\kappa\}$ shows that $\mathcal{I}$ is not κ-saturated, and this contradiction completes the proof

With these lemmas at our disposal we can now prove that if every normal ideal on κ satisfies Fodor's property, then there is no κ-saturated set $\mathcal{I}$ of normal ideals on κ such that $|\mathcal{I}| \leq \kappa$.

Proof that Theorem 6.2 (v) → (iv). Assume that every normal ideal on κ satisfies Fodor's property and let $\mathcal{I}$ be a set of normal ideals on κ such that $|\mathcal{I}| \leq \kappa$. We define sets $\mathcal{I}_0$, $\mathcal{I}_1$ and $\mathcal{I}_2$ as follows

$$\mathcal{I}_0 = \{I \in \mathcal{I} : I \text{ is nowhere } \kappa^+\text{-saturated}\},$$

$$\mathcal{I}_1 = \{I \in \mathcal{I} : \exists A_I \in I^+ \text{ s t } I \mid A_I \text{ is } \kappa^+\text{-saturated}\},$$

$$\mathcal{I}_2 = \{I \mid A_I : I \in \mathcal{I}_1\}$$

By Lemma 6 4 there exists a set $\{Y_\alpha : \alpha < \kappa^+\} \subseteq \mathcal{I}_0^+$ such that $Y_\alpha \cap Y_\beta \in \mathrm{NS}_\kappa$ whenever $\alpha < \beta < \kappa^+$. At most one Y_α can be of J-measure one for any single $J \in \mathcal{I}_2$ since $\mathrm{NS}_\kappa \subseteq J$. Hence we can choose $\gamma < \kappa^+$ such that $Y_\gamma \notin J^*$ for any $J \in \mathcal{I}_2$. Let $A = Y_\lambda$ and let $B = \kappa - A$. Then $A \in \mathcal{I}_0^+$, $B \in \mathcal{I}_2^+$ and $A \cap B = 0$.

Applying Lemma 6 5 to $\{J \mid B : J \in \mathcal{I}_2\}$ yields a pairwise disjoint partition $\{B_\xi : \xi < \kappa\}$ of B such that for each $\xi < \kappa$ $B_\xi \in \mathcal{I}_2^+ \subseteq \mathcal{I}_1^+$. Similarly, if we apply Lemma 6 4 to $\{I \mid A : I \in \mathcal{I}_0\}$ we obtain a pairwise disjoint partition $\{A_\xi : \xi < \kappa\}$ of A such that for each $\xi < \kappa$ $A_\xi \in \mathcal{I}_0^+$. But now $\{B_\xi \cup A_\xi : \xi < \kappa\}$ shows that $\mathcal{I}$ is not κ-saturated.

7. Dense sets and ω_1-complete ideals on ω_1

As previously remarked, our investigations of these regularity properties of κ-complete ideals on κ were inspired by Fodor's question as to whether or not NS_{ω_1} satisfies what we call Fodor's property (i e $(0, 2, \omega_1)$-regularity). This question was the central issue dealt with by Baumgartner et al [2], where they point out that some special features arise when considering the problem with respect to ω_1-complete ideals on ω_1 as opposed to κ-complete ideals on some larger successor cardinal κ. In this context they work with a condition that appears somewhat technical in [2] but is actually quite natural when isolated as in the following definition

Definition 7.1. A κ-complete ideal I on κ is said to have *a dense set of size* κ iff there exists $\mathcal{X} \subseteq I^+$ such that $|\mathcal{X}| = \kappa$ and such that for every $Y \in I^+$ there exists $X \in \mathcal{X}$ such that $X - Y \in I$

If I is a κ-complete ideal on κ, then we let $\mathcal{P}(\kappa)/I$ denote the Boolean algebra of subsets of κ mod I. Then I has a dense set of size κ (in the sense of Definition 7 1) iff $\mathcal{P}(\kappa)/I$ has a dense set of size κ (in the "forcing theoretic" sense). The result that we wish to both use and to generalize is the following

Lemma 7.2 (Baumgartner et al [2]) *Suppose I is a normal ideal on ω_1 such that there is no set $A \in I^+$ for which $I \mid A$ has a dense set of size ω_1. Then I satisfies Fodor's property*

The techniques developed in the previous section allow us to obtain the following strengthening of Lemma 7 2 This result was noticed independently by Balcar and Vojtáš [1] (via a rather different proof)

Theorem 7.3. *For any ω_1-complete ideal I on ω_1 the following are equivalent*

(i) *I fails to satisfy Fodor's property*

(ii) *$I \mid A$ has a dense set of size ω_1 for some set $A \in I^+$*

Proof. (ii)$\to$(i) Let $\{A_\alpha\ \alpha<\omega_1\}$ be a dense set of size ω_1 for $I \mid A$ If I satisfies Fodor's property, then there exists an I-$(0, 2, \omega_1)$-refinement $\{B_\alpha\ \alpha<\omega_1\} \subseteq I^+$ of $\{A_\alpha\ \alpha<\omega_1\}$ Clearly $\{B_\alpha\ \alpha<\omega_1\}$ is also a dense set of size ω_1 for $I \mid A$ For each $\alpha<\omega_1$ let $\{B^0_\alpha, B^1_\alpha\}$ be a pairwise disjoint partition of $B_\alpha \cap A$ into sets of positive I-measure and let $B=\bigcup\{B^0_\alpha\ \alpha<\omega_1\}$ Then $B \in (I \mid A)^+$ but $B'_\alpha \subseteq B_\alpha - B$ for every $\alpha<\omega_1$ so $\{B_\alpha\ \alpha<\omega_1\}$ is not a dense set for $I \mid A$

(i)$\to$(ii) Suppose that no $I \mid A$ has a dense set of size ω_1 for any $A \in I^+$ We show that I is $(0, 2, \omega_1)$-regular by appealing to Lemma 4 5 Suppose then that $\{A_\alpha \cdot \alpha<\omega_1\} \subseteq I^+$ is given Fix $\alpha<\omega_1$ and let $J=I \mid A_\alpha$ If J is nowhere ω_2-saturated, then J is $(0, 2, \omega_1)$-regular by Lemma 4 3 Otherwise $J \mid B$ is ω_2-saturated for some $B \in \mathcal{P}(A_\alpha) \cap J^+$ so by Theorem 3 3 we obtain a set $B_\alpha \in J^+$ such that $J \mid B_\alpha \cong N$ for some normal ideal N on ω_1 Clearly $N \mid A$ has no dense set of size ω_1 for any $A \in N^+$ so Lemma 7.2 guarantees that $J \mid B_\alpha$ is $(0, 2, \omega_1)$-regular Thus $\{A_\alpha\ \alpha<\omega_1\}$ has an I-$(0, 2, \omega_1)$-refinement as desired

A consequence of Theorem 5 1 is that an ω_1-complete ideal on ω_1 is regular iff it satisfies Fodor's property Theorem 7 2 yields a much easier proof of this result (for ω_1) since it is quite easy to see that a dense set $\{A_\alpha\ \alpha<\omega_1\}$ for I cannot have an I-$(0, \omega, \omega_1)$-refinement However, Theorem 4 1 does guarantee that a dense set for I has an I-$(0, \omega+1, \omega_1)$-refinement This yields the following rather striking "structure theorem" for dense sets Theorem 7 4 is due, in part, to Stan Wagon

Theorem 7.4. *Suppose I is an ω_1-complete ideal on ω_1 and I has a dense set of size ω_1 Then I has a dense set $\{B_\alpha\ \alpha<\omega_1\}$ such that if $\beta<\alpha$, then either $B_\beta \supseteq B_\alpha$ or $B_\beta \cap B_\alpha=0$*

Proof. Let $\{S_\alpha\ \alpha<\omega_1\} \subseteq I^+$ be a dense set for I and let $\{A_\alpha\ \alpha<\omega_1\}$ be the I-$(0, \omega+1, \omega_1)$-refinement of $\{S_\alpha\ \alpha<\omega_1\}$ guaranteed to exist by Theorem 4 1 We now inductively define a refinement $\{B_\alpha\ \alpha<\omega_1\} \subseteq I^+$ of $\{A_\alpha . \alpha<\omega_1\}$ as follows If B_β has been defined for every $\beta<\alpha$, then each point $\xi \in A_\alpha$ occurs in only finitely many B_β's for $\beta<\alpha$ (since intersections of type $\omega+1$ are empty) Hence there exists a set $B_\alpha \in \mathcal{P}(A_\alpha) \cap I^+$ and a set $s_\alpha \in [\alpha]^{<\omega}$ such that if $\beta<\alpha$ then $B_\beta \cap B_\alpha \neq 0$ iff $\beta \in S_\alpha$ iff $B_\beta \supseteq B_\alpha$ Deleting repetitions from $\{B_\alpha\ \alpha<\alpha\}$ yields the desired set

Corollary 7.5. *Suppose I is an ω_1-complete ideal on ω_1 and I has a dense set of size ω_1. Let $\mathrm{Seq} = \bigcup\{{}^n\omega_1 : n \in \omega\}$. Then I has a dense set $\{C_s : s \in \mathrm{Seq}\}$ satisfying the following*

(i) *If $s \in \mathrm{Seq}$ and $\varepsilon < \omega_1$, then $C_{s^\frown\varepsilon} \subseteq C_s$.*

(ii) *If $s \in \mathrm{Seq}$ and $\varepsilon_1 < \varepsilon_2 < \omega_1$, then $C_{s^\frown\varepsilon_1} \cap C_{s^\frown\varepsilon_2} = 0$.*

Proof. Let $\{B_\alpha : \alpha < \omega_1\}$ be a dense set for I as guaranteed to exist by the proof of Theorem 7.4 and, without loss of generality, assume that $B_0 = \omega_1$. If $0 < \alpha < \omega_1$, then $\{\beta < \omega_1 : B_\beta \supseteq B_\alpha\}$ is a finite subset of α containing 0. Hence, we can define a function $f : \omega_1 - \{0\} \to \omega_1$ by

$$f(\alpha) = \text{largest } \beta \text{ such that } B_\beta \supseteq B_\alpha$$

We begin by constructing a tree $\mathcal{T}_0$ having the following properties:

(i) the nodes of $\mathcal{T}_0$ are precisely the elements of $\{B_\alpha : \alpha < \omega_1\}$,

(ii) the immediate successors of a given node are pairwise disjoint subsets of the given node,

(iii) $\mathcal{T}_0$ is of height ω and is $\leqslant\omega_1$ branching.

Let $\omega_1 = B_0$ be the root of the tree. Proceeding inductively, suppose that the first n levels of the tree have been constructed and that B_β is a node on level n. Let the immediate successors of B_β be $\{B_\alpha : f(\alpha) = \beta\}$. Clearly each such B_α is a subset of B_β. To see that the successors are pairwise disjoint, suppose $f(\alpha_1) = \beta = f(\alpha_2)$ and $B_{\alpha_1} \cap B_{\alpha_2} \neq 0$. If $\beta < \alpha_1 < \alpha_2$, then $B_{\alpha_1} \supseteq B_{\alpha_2}$ and $\alpha_1 > \beta$ so $f(\alpha_2) = \alpha_1$ and this is a contradiction. Finally, we show that each B_α occurs on the tree (and this will also force the tree to have height ω). For the sake of contradiction suppose α is the least ordinal such that B_α does not occur on the tree. Then $\alpha > 0$ so $f(\alpha) < \alpha$. Thus $B_{f(\alpha)}$ occurs on the tree and hence, by our construction, B_α occurs as an immediate successor of $B_{f(\alpha)}$.

To complete the proof of Corollary 7.5 we modify $\mathcal{T}_0$ so that the resulting tree is exactly ω_1-branching. This modification is made possible by the following observation: the set of nodes of $\mathcal{T}_0$ having exactly ω_1 immediate successors is dense in $\mathcal{T}_0$. That is, if B_α is a node of $\mathcal{T}_0$, then the set of (eventual) successors of B_α in $\mathcal{T}_0$ is clearly a dense set for $I \mid B_\alpha$, and hence must be of cardinality ω_1. The claim now follows since a countably branching tree of height ω has only $\aleph_0$ nodes. Given this observation, we modify $\mathcal{T}_0$ level by level as follows. Level 0 is not changed. Suppose the first n levels have been altered and let B_α be an arbitrary node on level $n+1$. Choose B_β so that B_β is an eventual successor of B_α (in $\mathcal{T}_0$) and such that B_β has exactly ω_1 immediate successors. If B_β occurs on level k, then we let the new set of successors of B_α be the set of all B_γ's that occur on level $k+1$ and that are successors of B_α (in $\mathcal{T}_0$). Notice that the nodes of the resulting tree still constitute a dense set since each node of $\mathcal{T}_0$ contains (as a subset) at least one node of the resulting tree. This completes the proof.

In recent unpublished work, Laver has shown that $\mathrm{MA}_{\aleph_1}$ (i.e. Martin's axiom for $\aleph_1$, see [19]) implies that every uniform ultrafilter on ω_1 is regular. The results of this section allow us to prove the analogous result for ω_1-complete ideals on ω_1.

Theorem 7.6. *Assume* $\mathrm{MA}_{\aleph_1}$ *Then every ω_1-complete ideal on ω_1 is regular.*

Proof. Suppose J is a non-regular ω_1-complete ideal on ω_1 Then, by Corollary 5 4, there is a normal ideal N on ω_1 such that N does not satisfy Fodor's property. Hence, by Theorem 7 3, $N \mid A$ has a dense set of size ω_1 for some $A \in N^+$ Let $I = N \mid A$, and let $\{B_s \ s \in \mathrm{Seq}\}$ be a dense set for I as guaranteed to exist by Corollary 7 5. Our goal is to use $\mathrm{MA}_{\aleph_1}$ to (generically) obtain a regressive function f with domain $\{\alpha < \omega_1 \ \alpha$ is limit ordinal$\}$ such that for every $\alpha < \omega$ $\{s \in \mathrm{Seq} \ \alpha \notin f(B_s)\}$ is dense in the tree Seq.

Let P consist of all pairs $F = (f, A)$ such hat (1)–(3) are satisfied

(1) f is a finite regressive function and $\mathrm{dom}(f) \subseteq \{\alpha < \omega_1 : \alpha$ is a limit ordinal$\}$

(2) A is a finite subset of $\omega_1 \times \mathrm{Seq}$.

(3) If $(\alpha, s) \in A$ and $\xi \in \mathrm{dom}(f) \cap B_s$, then $f(\xi) \neq \alpha$

Let $P = \langle P, \leqslant \rangle$ where $(f_1, A_1) \leqslant (f_2, A_2)$ iff $f_1 \supseteq f_2$ and $A_1 \supseteq A_2$

For each $(\alpha, s) \in \omega_1 \times \mathrm{seq}$ let

$$D(\alpha, s) = \{(f, A) \in P \ \exists \xi < \omega_1 (\alpha, s^\frown \xi) \in A\}$$

Then $D(\alpha, s)$ is dense in P That is, given $(f, A) \in P$ and $(\alpha, s) \in \omega_1 \times \mathrm{Seq}$ we can choose $\xi < \omega_1$ such that $\mathrm{dom}(f) \cap B_{s^\frown \xi} = 0$ (because $B_{s^\frown \xi_1} \cap B_{s^\frown \xi_2} = 0$ for $\xi_1 \neq \xi_2$) Then $(f, A \cup \{(\alpha, s^\frown \xi)\} \in P \cap D(\alpha, s)$ and $(f, A \cup \{(\alpha, s^\frown \xi)\}) \leqslant (f, A)$

For each limit ordinal $\xi \in \omega_1 - \{0\}$ let

$$E(\xi) = \{(f, A) \in P. \ \xi \in \mathrm{dom}(f)\}$$

We claim that $E(\xi)$ is dense in P To see this, suppose $(f, A) \in P$ is given and assume (w l o.g) that $\xi \notin \mathrm{dom}(f)$ Choose $\alpha < \xi$ such that $(\alpha, s) \notin A$ for any $s \in \mathrm{Seq}$ Then $(f \cup \{(\xi, \alpha)\}, A) \in P \cap E(\xi)$ and $(f \cup \{(\xi, \alpha)\}, A) \leqslant (f, A)$

Before checking hat P satisfies the countable chain condition, we show that this will work

Let G be a P-generic set such that $G \cap D(\alpha, s) \neq 0$ and $G \cap E(\xi) \neq 0$ for every $(\alpha, s) \in \omega_1 \times \mathrm{Seq}$ and every limit ordinal $\xi \in \omega_1 - \{0\}$ Let $L = \{\xi < \omega_1 \cdot \xi > 0$ and ξ is a limit ordinal$\}$ and define $f \ L \to \omega_1$ by $f(\xi) = \alpha$ iff $(\xi, \alpha) \in f'$ for some $(f', A') \in G$ Then f is well-defined, regressive and $\mathrm{dom}(f) = L$ since $G \cap E(\xi) \neq 0$ for every $\xi \in L$ Since I is normal we have $L \in I^*$ so there exists $\alpha < \omega_1$ such that $f^{-1}(\alpha) \in I^+$. Since $\{B_s \ s \in \mathrm{Seq}\}$ is a dense set for I, there exists $s \in \mathrm{Seq}$ such that $B_s - f(\alpha) \in I$. Consider (α, s) Choose $(f', A') \in G \cap D(\alpha, s)$, and let $\xi \in \omega_1$ be such that $(\alpha, s\xi) \in A'$ Suppose that $\xi \in \mathrm{dom}(f) < B_{s^\frown \xi}$ and $f(\xi) = \alpha$ Then there exists $(f'', A'') \in G$ such that $(\xi, \alpha) \in f''$ But (f', A') and (f'', A'') are then incompatible since $\xi \in B_{s^\frown \xi}$ and $(\alpha, s^\frown \xi) \in A'$ and $f''(\xi) = \alpha$ Hence for every $\xi \in \mathrm{dom}(f) \cap B_{s^\frown \xi}$ we have $f(\xi) \neq \alpha$ Thus $B_{s^\frown \xi} \cap L \subseteq B_s - f^{-1}(\alpha)$ contradicting the fact that $B_s - f^{-1}(\alpha) \in I$.

Thus, to complete the proof, it suffices to show that P has the countable chain condition Suppose not and let $\{F_\alpha \ \alpha<\omega_1\}$ be an (indexed) set of pairwise incompatible elements of P where $F_\alpha=(f_\alpha, A_\alpha)$ By the usual Δ-system and thinning arguments (see [2] for a typical instance and references) we can assume the following

(i) $\{\operatorname{dom}(f_\alpha) \ \alpha<\omega_1\}$ forms a Δ-system with kernal D

(ii) $f_\alpha \mid D=f_\beta \mid D$ for every $\alpha, \beta<\omega_1$

(iii) $\{A_\alpha \ \alpha<\omega_1\}$ forms a Δ-system (in $\omega_1\times\mathrm{Seq}$)

(iv) There exists $n_0\in\omega$ such that if $(\alpha, s)\in A_\beta$ for some $\beta<\omega_1$, then length $(s)\leq n_0$

For $\beta_1<\beta_2$, F_{β_1} is incompatible with F_{β_2} iff either (a) or (b) holds

(a) $\exists(\alpha, s)\in A_{\beta_1}$ and $\xi\in\operatorname{dom}(f_{\beta_2})\cap B_s-D$ such that $f_{\beta_2}(\xi)=\alpha$

(b) $\exists(\alpha, s)\in A_{\beta_2}$ and $\xi\in\operatorname{dom}(f_{\beta_1})\cap B_s-D$ such that $f_{\beta_1}(\xi)=\alpha$

Define $g \ [\omega_1]^2\to 2$ by $g(\{\beta_1, \beta_2\})=0$ iff $\beta_1<\beta_2$ and (a) holds Using the partition relation $\omega_1\to(\omega+1, \omega)^2$ we consider two cases

Case 1 There is a set of order type $\omega+1$ homogeneous for 0 Let $F=(f, A)$ be the "$\omega+1$st point" of the homogeneous set Hence we get (reindexing) $\{F_n \ n\in\omega\}$ such that the following holds

$$\forall n\in\omega\ \exists(\alpha_n, s_n)\in A_n \quad \text{and} \quad \xi\in\operatorname{dom}(f)\cap B_{s_n} \text{ s t } f(\xi)=\alpha_n$$

Without loss of generality, we can assume that the same $\xi\in\operatorname{dom}(f)$ works for every $n\in\omega$ Now $\{s\in\mathrm{Seq} \ \xi\in B_s$ and length $(s)\leq n_0\}$ is finite so we can assume that $s_n=s$ for all $n\in\omega$ But of course we have a single α such that $\alpha=\alpha_n$ for all n since α_n is just $f(\xi)$ Thus (α, s) occurs in the kernal of the Δ-system for $\{A_\alpha \ \alpha<\omega_1\}$ Hence $(\alpha, s)\in A$ so $f(\xi)\neq\alpha$ This contradiction shows that Case 1 cannot occur

Case 2 There is a set of order type ω homogeneous for 1 Reindexing, we can assume that $\{F_n \ n\in\omega\}$ is the homogeneous set, and we have

$$\forall n>0\ \exists(\alpha_n, s_n)\in A_n \quad \text{and} \quad \xi\in\operatorname{dom}(f_0)\cap B_{s_n}-D \text{ s t } f_0(\xi)=\alpha_n$$

As before can assume that the same $\xi\in\operatorname{dom}(f_0)$ works for every $n>0$ and thus that $s_n=s$ for every $n\in\omega$ (using (iv) again) If $f_0(\xi)=\alpha$, then $\alpha_n=\alpha$ for all n so (α, s) is again in the kernal of the Δ-system for $\{A_\alpha \ \alpha<\omega_1\}$ so $f_0(\xi)\neq\alpha$ Hence Case 2 cannot occur either and so P has the countable chain condition as desired

We conclude with a final result that involves some restating of previous results but which serves as a reasonable summary concerning the regularity of ω_1-complete ideals on ω_1

Theorem 7.7. *The following assertions are equivalent, and all hold if* $\mathrm{MA}_{\aleph_1}$ *does*

(1) *No normal ideal on* ω_1 *has a dense set of size* ω_1

(2) *No* ω_1*-complete ideal on* ω_1 *has a dense set of size* ω_1

(3) *Every normal ideal on* ω_1 *is regular*

(4) *Every ω_1-complete ideal on ω_1 is regular.*

(5) *Every normal ideal on ω_1 satisfies Fodor's property*

(6) *Every ω_1-complete ideal on ω_1 satisfies Fodor's property*

(7) *If $\mathscr{I}$ is an ω_1-saturated set of normal ideals on ω_1, then $|\mathscr{I}|>\omega_1$*

(8) *If $\mathscr{I}$ is a set of normal ideals on ω_1 and $|\mathscr{I}|=\omega_1$, then there exists a set $X\subseteq\omega_1$ such that for every $I\in\mathscr{I}$ $X\notin(I\cup I^*)$*

Proof. The equivalence of the first seven follows fro previous results Clearly (7) implies (8) To see that (8) implies (1) suppose I is a normal ideal on ω_1 and $\{X_\alpha\ \alpha<\omega_1\}$ is a dense set for I Let $=\{I\,|\,X \quad <\omega_1\}$ Then $\mathscr{I}$ shows that (8) fails (The fact that (8) implies (1) is du Stan Wagon, and was noticed independently by Balcar and Vojt , [1])

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